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More on the Sixth Coefficient of the Matching Polynomial in Regular Graphs

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ABSTRACT

A matching set M in a graph G is a collection of edges of G such that no two edges from M share a vertex. The matching polynomial of G of order n is defined by $\mu(G,x) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^r \rho(G,r)x^{n-2r}$, where $\rho(G,r)$ is the number of matching sets of G with cardinality r and $p(G,0)$ is considered to be one. A graph that is characterized by its matching polynomial is said to be matching unique. In this paper, we consider some parameters related to the matching of regular graphs. We find the sixth coefficient of the matching polynomial of regular graphs. As a consequence, every cubic graph of order 10 is matching unique.

Keywords: Saturation number, matching polynomial and regular graph.

1. Introduction

A set of independent edges of a graph G is called a matching of G. A matching M is maximum if there is no matching in G with more edges than M. The cardinality of any maximum matching in G is called the matching number of G and is denoted by $\alpha'(G)$. If each vertex of G is incident with an edge of M, the matching M is called perfect. Only graphs of even order n can have a perfect matching and the size of such a matching is $\frac{n}{2}$. A matching M in G is maximal if no other matching in G contains it as a proper subset. The cardinality of any smallest maximal matching in G is the saturation number of G and is denoted by $s(G)$ (the same term, saturation number, is also used in the literature with a different meaning; we refer the reader to Faudree et al. (2009) for more information).

If a graph G has a maximum matching of size k, then any maximal matching has at least size $\frac{k}{2}$ (Biedl et al. (2004)). This implies that $s(G) \geq \frac{\alpha'(G)}{2}$. We recall that a set of vertices I is independent if no two vertices from I are adjacent. Clearly, the set of vertices that is not covered by a maximal matching is independent. This observation provides an obvious lower bound on saturation number of the graph G, i.e. $s(G) \geq \frac{(n-|I|)}{2}$ where G is graph of order n (Andova et al. (2015)).

An r-matching in a graph G is a set of r pairwise non-incident edges. The number of r-matchings in G is denoted by $\rho(G,r)$. The matching polynomial of G is defined by $\mu(G,x) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^r \rho(G,r)x^{n-2r}$, where n is the order of G and $p(G,0)$ is considered to be 1 (see Gutman (1979, 1983, 2006), Gutman and Harary (1983)). Two graphs are co-matching if their matching polynomials are equal. A graph that is characterized by its matching polynomial is said to be matching unique. Beezer and Farrell (1995) proved that if H is a regular graph of degree r on n vertices, and G is comatching with H, then G is also regular of degree r on n vertices. Also they have shown that if G is a graph that is comatching with a regular graph H, then G is a regular graph, and G and H have the same number of vertices, the same degree, the same girth, and the same number of cycles of minimal length. Ghorbani (2013) determined all graphs whose matching polynomials have at most five distinct zeros. As a consequence, he found new families of graphs which are determined by their matching polynomials.

Computation of matching polynomial is equivalent to computation of the number of k-matchings in the graph, for all k. The Petersen graph is one of the famous graph which is a symmetric non-planar cubic graph of order 10. Behmaram (2009) established a formula for the number of 4-matchings in triangular-free graph with respect to the number of vertices, edges, degrees and 4-cycles. Also he has proved that the Petersen graph is uniquely determined by its matching polynomial. The Petersen graph is one of the cubic graphs of order 10 and this is natural to study the matching polynomial of all of cubic graphs of order 10 which some of them are not triangular free. Vesalian and Asgari (2013) has obtained a formula for the number of 5-matchings in triangular-free and 4-cycle-free graph based on the number of vertices, edges, the degrees of vertices and the number of 5-cycles.

In the next section, after investigation of the saturation number and the matching number of cubic graphs of order 10, we establish a formula for the number of 5-matchings in regular graphs by using the method in Beezer (2006), Beezer and Farrell (1995). Using our result, we present the matching polynomial of cubic graphs of order 10 and conclude that every cubic graph of order ten is matching unique.

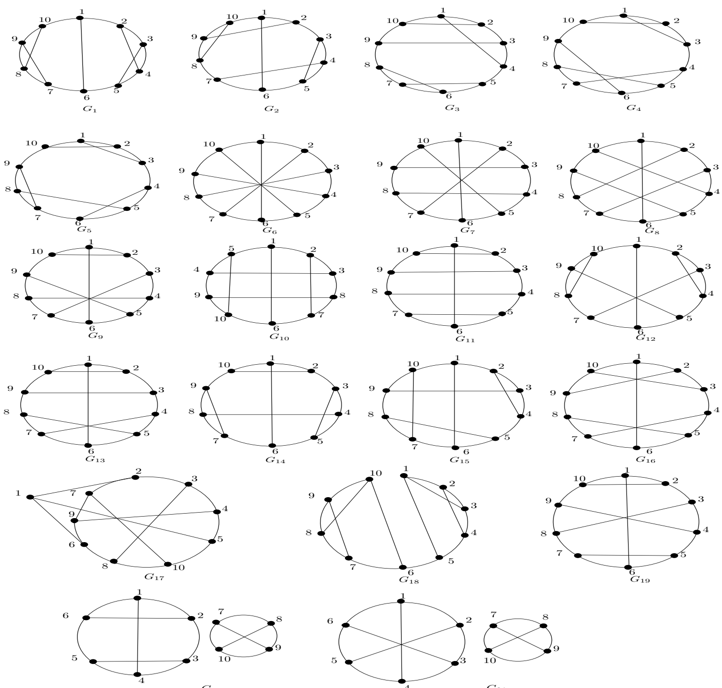


Figure 1: Cubic graphs of order 10.

2. Main results

First we consider the saturation number of cubic graphs of order 10 as a special kind of regular graphs. We recall that a cubic graph is a 3-regular graph. There are exactly 21 cubic graphs of order 10 given in Figure 1 (see Khosrovshahi et al. (2001)). Note that the graph G_{17} is the Petersen graph. The following observation presents the saturation number of cubic graphs with 10 vertices. Note that $s(G_{21}) = 5$.

Observation 2.1. *(i)* $s(G_i) = 3$ *for* $i = 1, 2, 5, 8, 12, 16, 17, 18$.

(ii) $s(G_i) = 4$ *for* $i = 3, 4, 6, 7, 9, 10, 11, 13, 14, 15, 19, 20$.

Since each 2-connected 3-regular graph has a perfect matching and a maximum matching in a disconnected graph consists of the union of maximum matchings in each of its components, so we can conclude that every cubic graph of order 10 has perfect matching. Thus for each $1 \leq i \leq 21$, $\alpha'(G_i) = 5$.

The domination polynomial and the edge cover polynomial of cubic graphs of order 10 has studied in Alikhani and Peng (2011) and Alikhani and Jahari (2014), respectively. It has proved that all cubic graphs of order 10 are determined uniquely by their edge cover polynomials, but this is not true for their domination polynomials. Here we consider the matching polynomial of regular graphs. Suppose that G is a regular graph of degree r with n vertices and $\Gamma_{m,k,i}$ is an isomorphism class containing i kind of spanning subgraphs of G, with m edges from G and k vertices of degree 1. Let $G_{m,k,i}$ denote a representative chosen from $\Gamma_{m,k,i}$ and $g_{m,k,i}$ is the cardinality of the set $\Gamma_{m,k,i}$ (Beezer and Farrell (1995)). Then the matching polynomial of graph G can be expressed as $\mu(G,x) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^m g_{m,2m,1} x^{n-2m}$. Note that $g_{m,2m,1}$ denotes the number of m-matchings of the graph G, for each m and that $g_{0,0,1} = 1$. The following theorem gives an approach for acquiring the coefficients of the matching polynomial of regular graphs.

Theorem 2.1. Beezer (1994) *Suppose that G is a regular graph of degree r on n vertices. Then for each combination of m, k and i^* there exist constants $a_{j,i} = a_{j,i}(n,r)$, that depend on G only through n and r, and constants a_i that are independent of G, so that*

$$g_{m,k,i^*} = \sum_{j=0}^{m-1} \sum_i a_{j,i}(n,r) g_{j,0,i} + \sum_i a_i g_{m,0,i}.$$

The following equations for $g_{1,2,1}, g_{2,2,1}, ..., g_{4,8,1}$ is obtained in Beezer (2006).

$$
\begin{aligned}
g_{1,2,1} &= \tfrac{nr}{2},\\
g_{2,2,1} &= \tfrac{n(r-1)r}{2},\\
g_{2,4,1} &= \tfrac{nr}{8}(nr-4r+2),\\
g_{3,2,1} &= \tfrac{n(r-1)^2r}{2} - 3g_{3,0,1},\\
g_{3,3,1} &= \tfrac{n(r-2)(r-1)r}{6},\\
g_{3,4,1} &= \tfrac{n(r-1)r(nr-6r+4)}{4} + 3g_{3,0,1},\\
g_{3,6,1} &= \tfrac{nr}{48}(n^2r^2 - 12nr^2 + 40r^2 + 6nr - 48r + 16) - g_{3,0,1},\\
g_{4,1,1} &= (3r-6)g_{3,0,1},\\
g_{4,2,1} &= \big(\tfrac{nr}{2} - 3r + 3\big)g_{3,0,1},\\
g_{4,2,2} &= \tfrac{n(r-1)^3r}{2} + (-6r+9)g_{3,0,1} - 4g_{4,0,1},\\
g_{4,3,1} &= \tfrac{n(r-2)(r-1)^2r}{2} + (-6r+12)g_{3,0,1},\\
g_{4,4,1} &= \tfrac{n(r-1)^2r(nr-8r+6)}{4} + \big(-\tfrac{3nr}{2} + 18r - 21\big)g_{3,0,1} + 4g_{4,0,1},\\
g_{4,4,2} &= \tfrac{n(r-3)(r-2)(r-1)r}{24},\\
g_{4,4,3} &= \tfrac{n(r-1)^2r(nr-9r+8)}{8} + (6r-9)g_{3,0,1} + 2g_{4,0,1},\\
g_{4,5,1} &= \tfrac{n(r-2)(r-1)r(nr-8r+6)}{12} + (3r-6)g_{3,0,1},\\
g_{4,6,1} &= \tfrac{n(r-1)r(n^2r^2-16nr^2+72r^2+10nr-104r+40)}{16} + \big(\tfrac{3nr}{2} - 21r + 24\big)g_{3,0,1} - 4g_{4,0,1},\\
g_{4,8,1} &= \tfrac{nr}{384}\big(n^3r^3 - 24n^2r^3 + 208nr^3 - 672r^3 + 12n^2r^2 - 240nr^2 + 1344r^2 + 76nr\\
&\quad - 960r + 240\big) + \big(-\tfrac{nr}{2} + 6r - 6\big)g_{3,0,1} + g_{4,0,1}.
\end{aligned}
$$

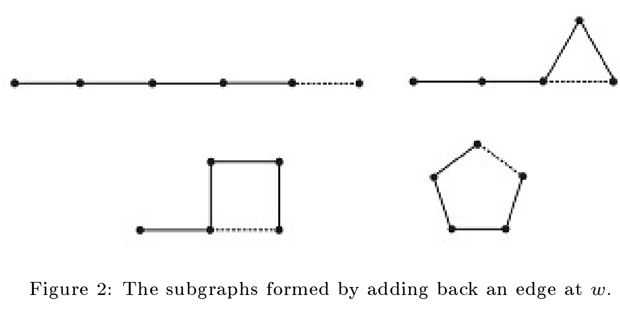
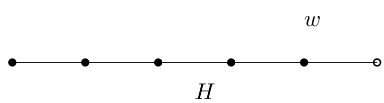


Figure 2: The subgraphs formed by adding back an edge at w.

We follow approach in Beezer (2006) to derive a formula for the sixth coefficient of the matching polynomial in regular graphs, i.e., $g_{5,10,1}$. We start with possible subgraphs on five edges or less (see Figures 3 and 4) and determine the subgraphs with 5 edges and a vertex of degree 1 (see Wilson and Read (2005)). Then remove the edge incident to degree 1 vertex and call other endpoint w. We identify vertices isomorphic to w and then add back a single edge, attaching one end at vertices like w. Finally, we determine the types of subgraphs formed and the amount of overcounting. Note that the subgraphs with no degree 1 vertices are free variables. For example, Let H be a subgraph of regular graph G with 5 edges. Remove an edge incident to a vertex of degree 1. Label other endpoint w. In the path on 4 edges that remains, there is one other vertex like w. So there are two vertices like w and $r-1$ ways to attach back an edge at w, considering all vertices as possibilities for the other end of the new edge (see Figure 2).



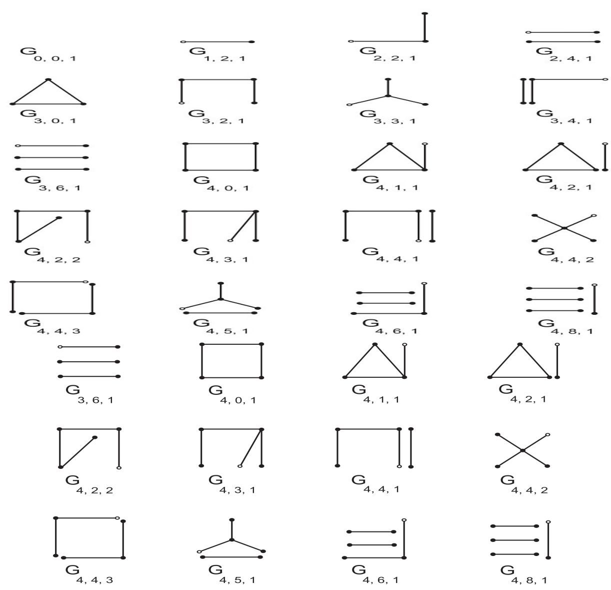


Figure 3: The possible subgraphs with ≤ 4 edges. Vertex w is adjacent to open-circle vertex.

So we have $2(r-1)g_{4,2,2} = 2g_{5,2,2} + 2g_{5,1,2} + 2g_{5,1,1} + 10g_{5,0,1}$.

By using this process for every subgraph of regular graph G with 5 edges and a vertex of degree 1, we have the following system of linear equations.

$$
\begin{aligned}
4(r-2)g_{4,0,1} &= 1g_{5,1,1} + 2g_{5,0,2},\\
1(r-1)g_{4,1,1} &= 1g_{5,1,2} + 4g_{5,0,2},\\
(n-4)rg_{4,0,1} &= 2g_{5,2,1} + 1g_{5,1,1},\\
2(r-1)g_{4,2,2} &= 2g_{5,2,2} + 2g_{5,1,2} + 2g_{5,1,1} + 10g_{5,0,1},\\
2(r-1)g_{4,2,1} &= 2g_{5,2,3} + 1g_{5,1,2},\\
2(r-2)g_{4,1,1} &= 2g_{5,2,4} + 4g_{5,0,2},\\
1(r-3)g_{4,1,1} &= 2g_{5,2,5},\\
2(r-2)g_{4,2,2} &= 2g_{5,3,1} + 2g_{5,1,1} + 2g_{5,2,4},\\
2(r-1)g_{4,3,1} &= 2g_{5,3,2} + 2g_{5,1,2} + 2g_{5,2,4} + 2g_{5,1,1},\\
3(r-2)g_{4,2,1} &= 1g_{5,3,3} + 1g_{5,1,2},\\
2(r-1)g_{4,4,1} &= 2g_{5,4,1} + 2g_{5,3,3} + 8g_{5,2,1} + 2g_{5,2,2},\\
4(r-1)g_{4,4,3} &= 2g_{5,4,2} + 6g_{5,2,3} + 2g_{5,2,2} + 1g_{5,3,1},\\
1(r-2)g_{4,3,1} &= 4g_{5,4,3} + 2g_{5,2,4},\\
1(r-3)g_{4,3,1} &= 3g_{5,4,4} + 2g_{5,2,5},\\
(n-5)rg_{4,2,1} &= 4g_{5,4,5} + 2g_{5,2,3} + 1g_{5,3,3},\\
3(r-1)g_{4,5,1} &= 1g_{5,5,1} + 2g_{5,3,3} + 1g_{5,3,1},\\
2(r-2)g_{4,4,3} &= 3g_{5,5,2} + 1g_{5,3,1} + 2g_{5,4,3},\\
1(r-4)g_{4,4,2} &= 5g_{5,5,3},\\
1(r-3)g_{4,5,1} &= 4g_{5,6,1} + 1g_{5,4,4},\\
2(r-1)g_{4,6,1} &= 2g_{5,6,2} + 6g_{5,4,5} + 2g_{5,4,1},\\
4(r-1)g_{4,6,1} &= 4g_{5,6,3} + 2g_{5,4,1} + 1g_{5,5,1} + 2g_{5,4,2},\\
1(r-2)g_{4,6,1} &= 3g_{5,7,1} + 1g_{5,5,1},\\
8(r-1)g_{4,8,1} &= 2g_{5,8,1} + 2g_{5,6,2},\\
(n-8)rg_{4,8,1} &= 10g_{5,10,1} + 2g_{5,8,1}.
\end{aligned}
$$

Now we can conclude that

$$
\begin{aligned}
g_{5,1,1} &= (4r-8)g_{4,0,1} - 2g_{5,0,2},\\
g_{5,1,2} &= (3r^2-9r+6)g_{3,0,1} - 4g_{5,0,2},\\
g_{5,2,1} &= \left(\tfrac{nr}{2} - 4r + 4\right)g_{4,0,1} + g_{5,0,2},\\
g_{5,2,2} &= \tfrac{n(r-1)^4 r}{2} + (-9r^2+24r-15)g_{3,0,1} + (-8r+12)g_{4,0,1} - 5g_{5,0,1} + 6g_{5,0,2},\\
g_{5,2,3} &= \left(\tfrac{nr^2-9r^2-nr+21r-12}{2}\right)g_{3,0,1} + 2g_{5,0,2},\\
g_{5,2,4} &= (3r^2-12r+12)g_{3,0,1} - 2g_{5,0,2},\\
g_{5,2,5} &= \left(\tfrac{3r^2-15r+18}{2}\right)g_{3,0,1},\\
g_{5,3,1} &= \tfrac{n(r-2)(r-1)^3 r}{2} + (-9r^2+33r-30)g_{3,0,1} + (-8r+16)g_{4,0,1} + 4g_{5,0,2},\\
g_{5,3,2} &= \tfrac{n(r-2)(r-1)^3 r}{2} + (-12r^2+39r-30)g_{3,0,1} + (-4r+8)g_{4,0,1} + 8g_{5,0,2},\\
g_{5,3,3} &= \left(\tfrac{3nr^2}{2} - 12r^2 - 3nr + 36r - 24\right)g_{3,0,1} + 4g_{5,0,2},\\
g_{5,4,1} &= \tfrac{n(r-1)^3 r(nr-10r+8)}{4} + \left(-3nr^2 + 39r^2 + \tfrac{9nr}{2} - 99r + 60\right)g_{3,0,1} + (-2nr+28r-32)g_{4,0,1} +\\
&\quad 5g_{5,0,1} - 14g_{5,0,2},\\
g_{5,4,2} &= \tfrac{n(r-1)^3 r(nr-12r+12)}{4} + \left(\tfrac{-3nr^2+78r^2+3nr-204r+132}{2}\right)g_{3,0,1} + (16r-24)g_{4,0,1} + 5g_{5,0,1} - 14g_{5,0,2},\\
g_{5,4,3} &= \tfrac{n(r-2)^2(r-1)^2 r}{8} + (-3r^2+12r-12)g_{3,0,1} + g_{5,0,2},\\
g_{5,4,4} &= \tfrac{n(r-3)(r-2)(r-1)^2 r}{6} + (-3r^2+15r-18)g_{3,0,1},\\
g_{5,4,5} &= \left(\tfrac{n^2r^2-16nr^2+72r^2+14nr-144r+72}{8}\right)g_{3,0,1} - 2g_{5,0,2},\\
g_{5,5,1} &= \tfrac{n(r-2)(r-1)^2 r(nr-10r+8)}{4} + (-3nr^2+42r^2+6nr-132r+96)g_{3,0,1} + (8r-16)g_{4,0,1} - 12g_{5,0,2},\\
g_{5,5,2} &= \tfrac{n(r-2)(r-1)^2 r(nr-12r+12)}{12} + (9r^2-33r+30)g_{3,0,1} + (4r-8)g_{4,0,1} - 2g_{5,0,2},\\
g_{5,5,3} &= \tfrac{n(r-4)(r-3)(r-2)(r-1)r}{120},\\
g_{5,6,1} &= \tfrac{n(r-3)(r-2)(r-1)r(nr-10r+8)}{48} + \left(\tfrac{6r^2-30r+36}{4}\right)g_{3,0,1},\\
g_{5,6,2} &= \tfrac{n(r-1)^2 r(n^2r^2-20nr^2+112r^2+14nr-176r+72)}{16} + \left(\tfrac{-3n^2r^2+84nr^2-696r^2-90nr+1584r-888}{8}\right)g_{3,0,1}\\
&\quad + (2nr-32r+36)g_{4,0,1} - 5g_{5,0,1} + 20g_{5,0,2},\\
g_{5,6,3} &= \tfrac{n(r-1)^2 r(n^2r^2-21nr^2+126r^2+16nr-216r+96)}{16} + \left(\tfrac{18nr^2-252r^2-24nr+714r-444}{4}\right)g_{3,0,1}+\\
&\quad (nr-28r+36)g_{4,0,1} - 5g_{5,0,1} + 17g_{5,0,2},\\
g_{5,7,1} &= \tfrac{n(r-2)(r-1)r(n^2r^2-20nr^2+112r^2+14nr-176r+72)}{48} + \left(\tfrac{3}{2}nr^2 - 21r^2 - 3nr + 66r - 48\right)g_{3,0,1}+\\
&\quad (-4r+8)g_{4,0,1} + 4g_{5,0,2},\\
g_{5,8,1} &= \tfrac{n(r-1)r(n^3r^3-30n^2r^3+328nr^3-1344r^3+18n^2r^2-444nr^2+3072r^2+160nr-2448r+672)}{96}\\
&\quad + \left(\tfrac{3n^2r^2-100nr^2+888r^2+106nr-1968r+1080}{8}\right)g_{3,0,1} + (-2nr+36r-40)g_{4,0,1} + 5g_{5,0,1} - 20g_{5,0,2},
\end{aligned}
$$

$$g_{5,10,1} = \frac{nr}{3840}\Big(n^4r^4 - 40n^3r^4 + 640n^2r^4 - 4960nr^4 + 16128r^4 + 20n^3r^3 - 720n^2r^3 + 9440nr^3 - 46080r^3 + 220n^2r^2 - 6400nr^2 + 51840r^2 + 1520nr - 26880r + 5376\Big) + \Big(\frac{-5n^2r^2+140nr^2-1080r^2-130nr+2160r-1080}{40}\Big)g_{3,0,1} + \Big(\frac{nr}{2} - 8r + 8\Big)g_{4,0,1} - g_{5,0,1} + 4g_{5,0,2}.$$

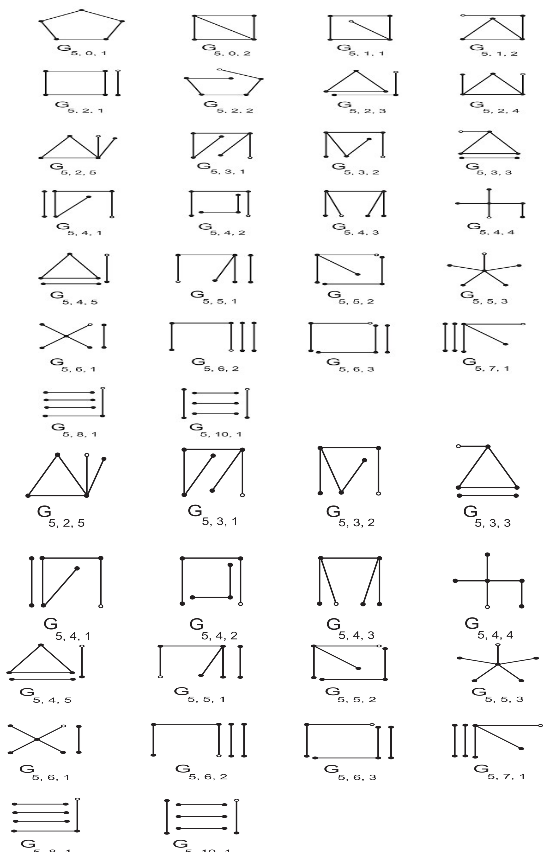


Figure 4: The possible subgraphs with 5 edges (vertex w is adjacent to open-circle vertex.

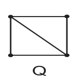


Figure 5: The subgraph Q.

The following theorem gives the matching polynomial of cubic graphs of order 10 as a specific kind of regular graphs.

Note that for every cubic graph G of order 10, $\alpha'(G) = 5$ and so it is enough to obtain $g_{m,2m,1}$ for every $m \leq 5$.

Theorem 2.2. *Let G be a cubic graph of order* 10*. Then the matching polynomial of G is*

$$\mu(G,x) = x^{10} - 15x^8 + 75x^6 - 145x^4 + 90x^2 - 18 + (x^4 - 3x^2 + 3)g_{3,0,1} + (x^2+1)g_{4,0,1} + g_{5,0,1} - 4g_{5,0,2}$$

where $g_{3,0,1}$, $g_{4,0,1}$ and $g_{5,0,1}$ are the number of circuits on 3, 4 *and* 5 *vertices, respectively and $g_{5,0,2}$ is the number of subgraph Q, where Q is the graph in Figure 5.*

Proof. It is sufficient to compute the coefficients of matching polynomial of G. We apply the formulas for $g_{m,2m,1}$ where $0 \leq m \leq 5$. We have $g_{1,2,1} = 15$, $g_{2,4,1} = 75$, $g_{3,6,1} = 145 - g_{3,0,1}$, $g_{4,8,1} = 90 - 3g_{3,0,1} + g_{4,0,1}$ and

$$g_{5,10,1} = 18 - 3g_{3,0,1} - g_{4,0,1} - g_{5,0,1} + 4g_{5,0,2}.$$

Now the result follows.

Behmaram proved that triangular-free 3-regular graphs with ten vertices are matching unique and so cubic graphs $G_6, G_7, G_8, G_{10}, G_{16}$ and G_{17} are matching unique (see Behmaram (2009)). In the following theorem, we show that every cubic graph of order 10 is matching unique.

Theorem 2.3. *Every cubic graph of order ten is matching unique.*

Proof. Suppose that $G_1, G_2, ..., G_{21}$ denote the cubic graphs of order 10 as shown in Figure 1. By replacing $g_{3,0,1}$, $g_{4,0,1}$, $g_{5,0,1}$ and $g_{5,0,2}$ for the cubic graphs with ten vertices in Theorem 2.2, we have

$$\mu(G_1,x) = x^{10} - 15x^8 + 75x^6 - 141x^4 + 80x^2 - 8,$$
$$\mu(G_2,x) = x^{10} - 15x^8 + 75x^6 - 143x^4 + 86x^2 - 6,$$
$$\mu(G_3,x) = x^{10} - 15x^8 + 75x^6 - 142x^4 + 84x^2 - 7,$$
$$\mu(G_4,x) = x^{10} - 15x^8 + 75x^6 - 143x^4 + 90x^2 - 10,$$

$$\mu(G_5, x) = x^{10} - 15x^8 + 75x^6 - 141x^4 + 80x^2 - 6,$$
$$\mu(G_6, x) = x^{10} - 15x^8 + 75x^6 - 145x^4 + 95x^2 - 13,$$
$$\mu(G_7, x) = x^{10} - 15x^8 + 75x^6 - 145x^4 + 93x^2 - 9,$$
$$\mu(G_8, x) = x^{10} - 15x^8 + 75x^6 - 145x^4 + 92x^2 - 8,$$
$$\mu(G_9, x) = x^{10} - 15x^8 + 75x^6 - 144x^4 + 89x^2 - 8,$$
$$\mu(G_{10}, x) = x^{10} - 15x^8 + 75x^6 - 145x^4 + 95x^2 - 11,$$
$$\mu(G_{11}, x) = x^{10} - 15x^8 + 75x^6 - 143x^4 + 87x^2 - 7,$$
$$\mu(G_{12}, x) = x^{10} - 15x^8 + 75x^6 - 143x^4 + 84x^2 - 6,$$
$$\mu(G_{13}, x) = x^{10} - 15x^8 + 75x^6 - 144x^4 + 91x^2 - 8,$$
$$\mu(G_{14}, x) = x^{10} - 15x^8 + 75x^6 - 142x^4 + 81x^2 - 6,$$
$$\mu(G_{15}, x) = x^{10} - 15x^8 + 75x^6 - 144x^4 + 90x^2 - 9,$$
$$\mu(G_{16}, x) = x^{10} - 15x^8 + 75x^6 - 145x^4 + 96x^2 - 12,$$
$$\mu(G_{17}, x) = x^{10} - 15x^8 + 75x^6 - 145x^4 + 90x^2 - 6,$$
$$\mu(G_{18}, x) = x^{10} - 15x^8 + 75x^6 - 141x^4 + 84x^2 - 4,$$
$$\mu(G_{19}, x) = x^{10} - 15x^8 + 75x^6 - 143x^4 + 85x^2 - 7,$$
$$\mu(G_{20}, x) = x^{10} - 15x^8 + 75x^6 - 139x^4 + 78x^2 - 12,$$
$$\mu(G_{21}, x) = x^{10} - 15x^8 + 75x^6 - 141x^4 + 90x^2 - 18.$$

Assume that H is co-matching with $G \in \{G_1, G_2, ..., G_{21}\}$, so H is also a cubic graph with 10 vertices. By comparing the matching polynomials of $G_1, G_2, ..., G_{21}$, we conclude that none of these polynomials are equal. Therefore G is matching unique.

3. Conclusion

In this paper, after investigation of the saturation number and the matching number of cubic graphs of order 10, we established a formula for the number of 5-matchings in regular graphs by using the method in Beezer (2006), Beezer and Farrell (1995). Using our result, we presented the matching polynomial of cubic graphs of order 10 and concluded that every cubic graph of order ten is matching unique.

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